15.6 TRIPLE INTEGRALS.

$$f: E \subset \mathbb{R}^3 \to \mathbb{R}$$
$$(x,y,z) \to w = f(x,y,z).$$

I) E is a rectangular box:

Go TO Stewart's slides.

- Definition of Riemann Sums. (Stewart).

Definition of Triple Integral over E (rectangular box).

$$\iiint_E f(x,y,z)\, dV =$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\right) \Delta V$$

Where $\Delta V = \Delta x \Delta y \Delta z$.

- Fubini's Theorem for triple Integrals:

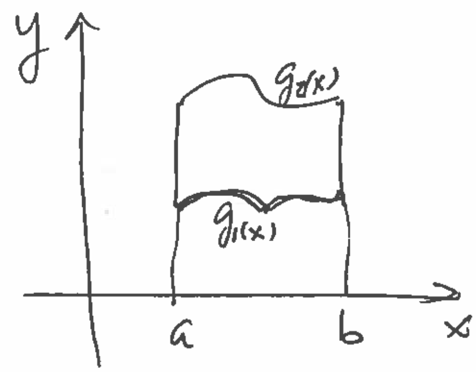
$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx.$$

TRIPLE INTEGRALS General Regions

Type I

2-D: $$\iint_D f(x,y)\,dA = \int_{a}^{b}\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

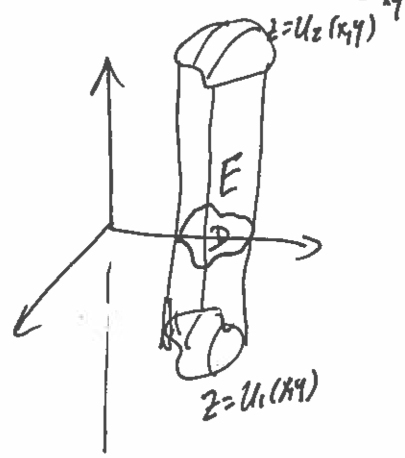
$x \qquad y$



Type I

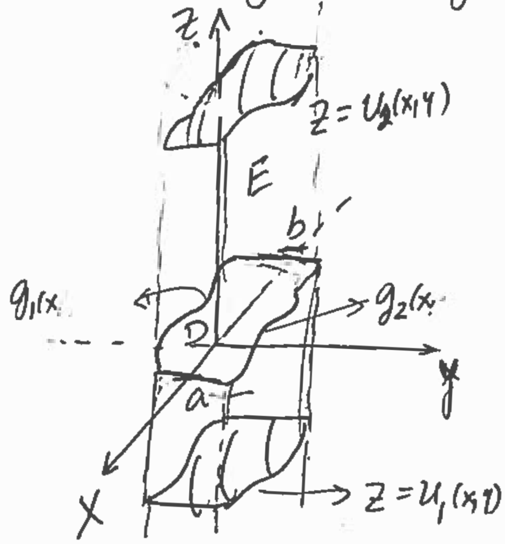
3-D: $$I = \iiint_E f(x,y,z)\,dV = \iint_{D_{xy}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

z



Two ways to obtain I.

1) E is type I region and its projection D is also a type I region in the xy-plane.



$$I=\iiint_E f(x,y,z)\,dV=\iint_{D_{xy}}\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA$$
$$\quad\quad\quad\quad\quad\quad\quad z$$

$$=\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$
$$x \quad y \quad z$$

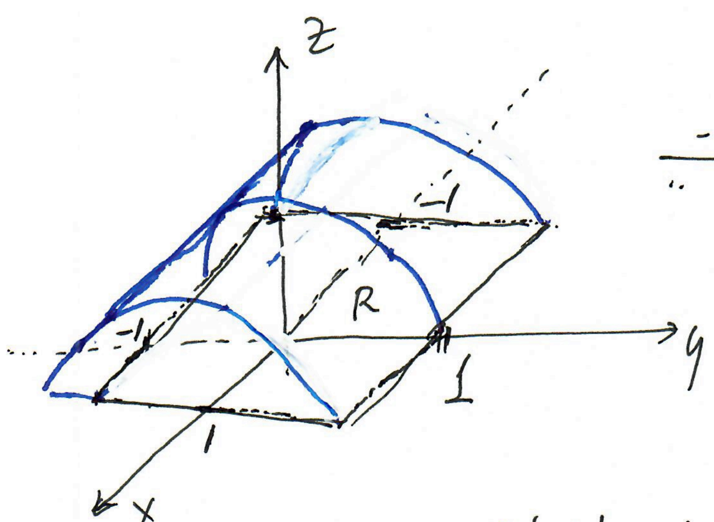
Ask for other possibilities and show book's graphs.

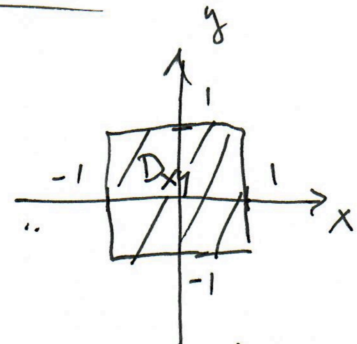
Compute Volume of Solid region R bounded by parabolic cylinder

$$z = 1 - y^2$$

above xy-plane and between planes

$$x = 1, \quad x = -1$$





Type 1 region:

$$\iiint_R dv = \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1-y^2} dz\,dy\,dx$$

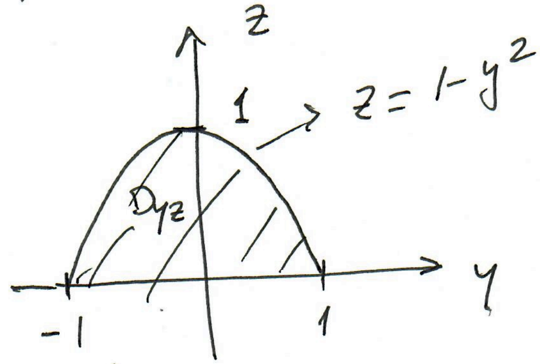
$$\qquad\qquad x \quad y \quad z$$

Type 2 region:

$$\iiint_R dv = \iint_{D_{yz}} \int_{-1}^{1} dx\,dz\,dy$$

$$\qquad\qquad\qquad x$$

In the yz-plane



$$= \int_{-1}^{1} \int_{0}^{1-y^2} \int_{-1}^{1} dx\, dz\, dy$$

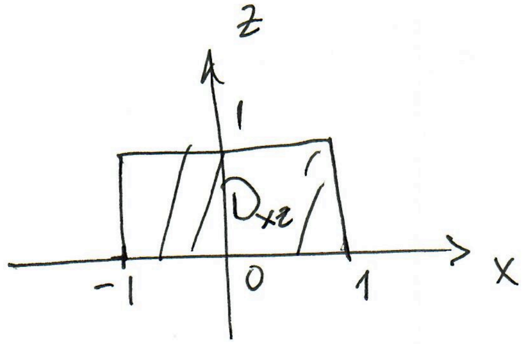
$y \quad z$

$z = +1 - y^2$

$y = \pm\sqrt{1-z}$

<u>Type 3 region:</u>

$$\iint_{D_{xz}} \int_{y=-\sqrt{1-z}}^{y=\sqrt{1-z}} dy\, dx\, dz$$



$$\int_{-1}^{1} \int_{-1}^{1} y \Big|_{-\sqrt{1-z}}^{\sqrt{1-z}} dx\, dz.$$

$z \quad x$

$$= \int_{-1}^{1} \int_{-1}^{1} \left(\sqrt{1-z} + \sqrt{1-z}\right) dx\, dz$$

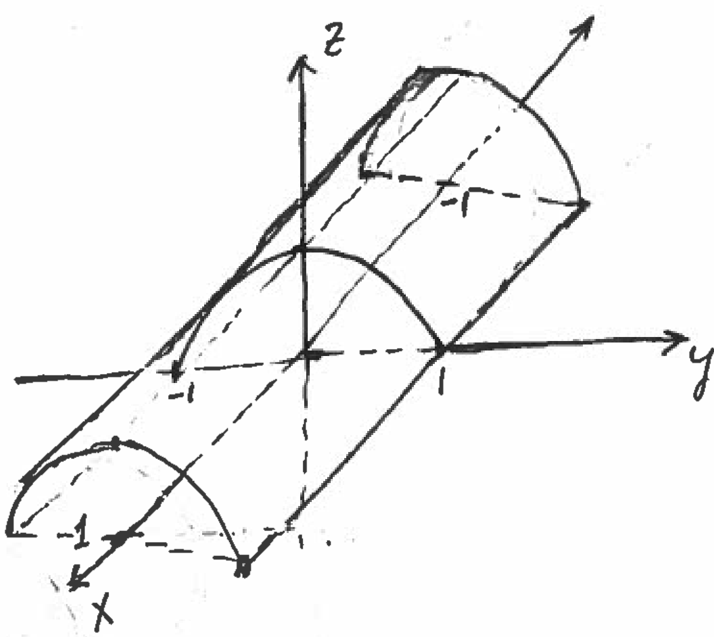
15.6

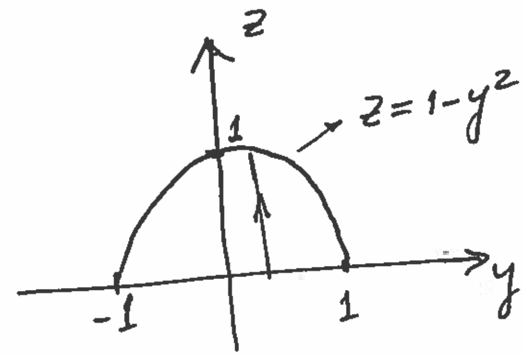
Triple Integrals

Example (not in book)

$$I = \iiint_E x^2 e^y \, dV,$$

E solid bounded by parabolic cylinder $z = 1 - y^2$, above xy-plane and between planes $x = 1$, $x = -1$.



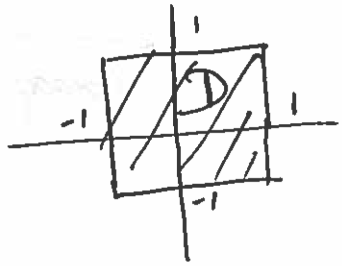


Type I region: $E = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D,\ 0 \le z \le 1 - y^2\}$

Where D is the projection of E into the xy-plane

In our case,

$$D = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1,\ -1 \le y \le 1\}$$



Then,

$$I = \iiint\limits_{E} x^2 e^y \, dv = \iint\limits_{\substack{D \\ (x,y)}} \left[\int_{\substack{0 \\ z(x,y)}}^{1-y^2} x^2 e^y \, dz \right] dA$$

$$= \int_{\substack{-1 \\ x}}^{1} \int_{\substack{-1 \\ y}}^{1} \int_{\substack{0 \\ z}}^{1-y^2} x^2 e^y \, dz \, dy \, dx = \int_{-1}^{1} \int_{-1}^{1} x^2 e^y z \Big|_{z=0}^{z=1-y^2} dy \, dx$$

$$= \int_{-1}^{1} \int_{-1}^{1} x^2 (1-y^2) e^y \, dy \, dx = \int_{-1}^{1} \int_{-1}^{1} (x^2 e^y - x^2 y^2 e^y) \, dy \, dx$$

(*)
$$= \int_{-1}^{1} x^2 e^y \Big|_{-1}^{1} dx - \int_{-1}^{1} x^2 (2 - 2y + y^2) e^y \Big|_{-1}^{1} dx =$$

$$= \int_{-1}^{1} x^2 (e - e^{-1}) \, dx - \int_{-1}^{1} x^2 \left[(\cancel{2} - \cancel{2} + 1^2) e - (2 + 2 + (-1)^2) e^{-1} \right] dx$$

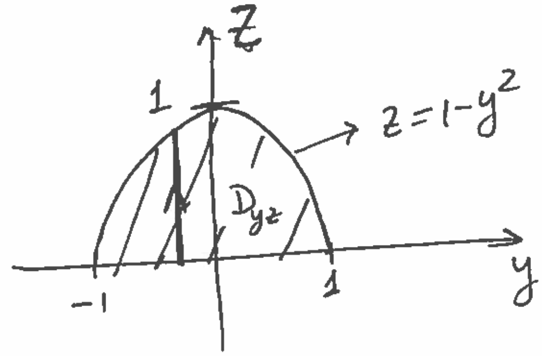
$$= (e - e^{-1}) \int_{-1}^{1} x^2 dx - (e - 5e^{-1}) \int_{-1}^{1} x^2 dx = 4e^{-1} \int_{-1}^{1} x^2 dx$$

$$= 4e^{-1} \frac{x^3}{3} \Big|_{-1}^{1} = 4e^{-1} \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{8}{3e}$$

(*) $\int y^2 e^y dy \overset{\text{I.P.P Twice}}{=\!=} (2 - 2y + y^2) e^y$

Domain of
Example in Triple Integrals used in a double integral

$$I = \iint_{D_{yz}} yz\, dA \qquad D_{xy} = \{(y,z) : -1 \le y \le 1,\ 0 \le z \le 1-y^2\}$$



type I
in the
yz-plane

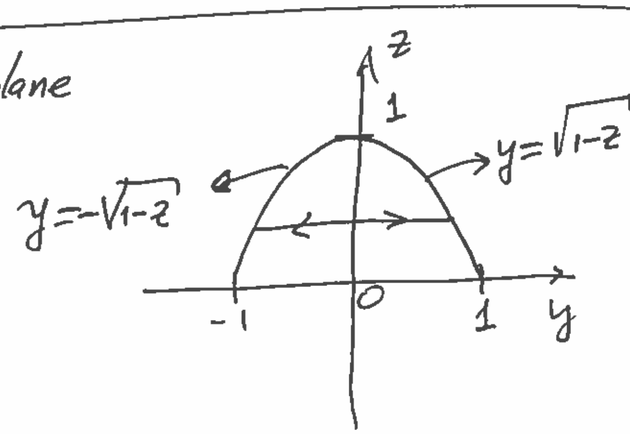
$$I = \int_{-1}^{1}\int_{0}^{1-y^2} yz\, dz\, dy = \int_{-1}^{1} y \frac{z^2}{2}\Big|_0^{1-y^2} dy$$

$$= \frac{1}{2}\int_{-1}^{1} y(1-y^2)^2\, dy = \frac{1}{2}\int_{-1}^{1}(y - 2y^3 + y^5)\, dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2}{4}y^4 + \frac{y^6}{6}\right]\Big|_{-1}^{1} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right)\right] = 0$$

As a type II in the yz-plane

$$I = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} yz\, dy\, dz$$



As a type 2 region:

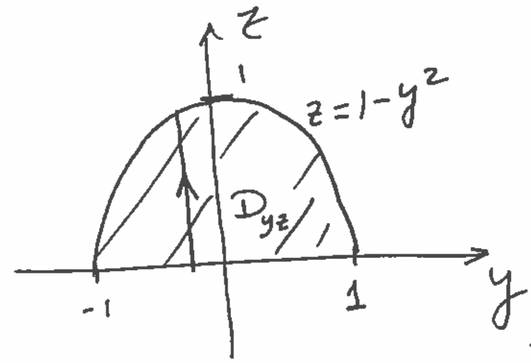
$$E = \{ (x,y,z) \in \mathbb{R}^3 : (y,z) \in D_{yz},\ -1 \le x \le 1 \}$$

Projection into yz-plane

$$I = \iint_{D_{yz}} \left(\int_{-1}^{1} x^2 e^y \, dx \right) dA$$

$$= \int_{-1}^{1} \int_{0}^{1-y^2} \int_{-1}^{1} x^2 e^y \, dx \, dz \, dy$$

$y \quad z \quad x$



$$D_{yz} = \left\{ (y,z) : \begin{array}{l} -1 \le y \le 1 \\ 0 \le z \le 1 - y^2 \end{array} \right\}$$

As a type 3 region:

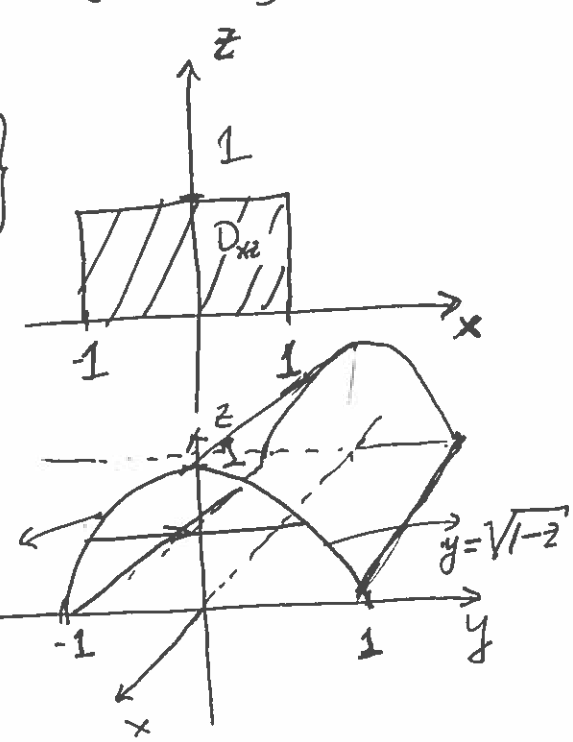
Projection of E into xz-plane

$$E = \{ (x,y,z) \in \mathbb{R}^3 : (x,z) \in D_{xz},\ -\sqrt{1-z} \le y \le \sqrt{1-z} \}$$

$$I = \iint_{D_{xz}} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} x^2 e^y \, dy \qquad D_{xz} = \left\{ (x,z) : \begin{array}{l} -1 \le x \le 1 \\ 0 \le z \le 1 \end{array} \right\}$$

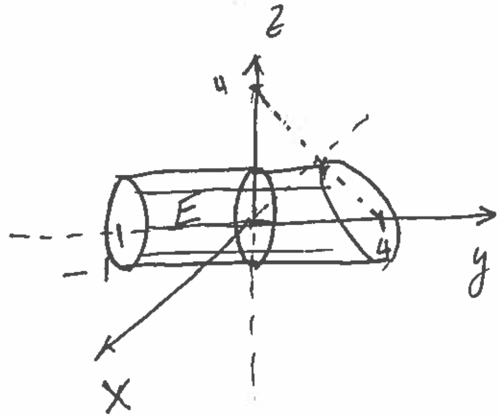
$$= \int_{-1}^{1} \int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} x^2 e^y \, dy \, dz \, dx$$

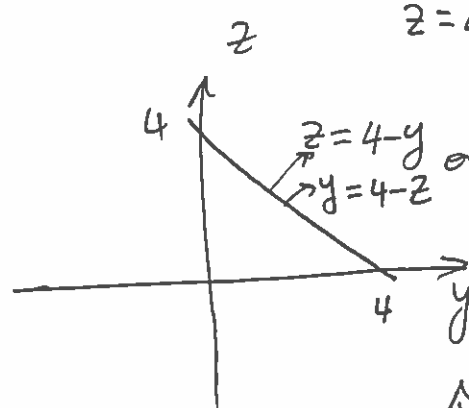
$x \quad z \quad y$



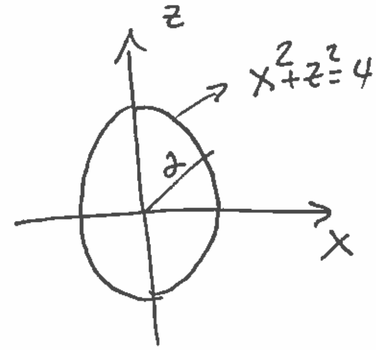
15.6 #22) Vol. of Solid enclosed by

Cylinder: $x^2+z^2=4$, plane 1: $y=-1$, plane 2: $y+z=4$ or
$y=4-z$.
$z=4-y$





$$\text{Vol } E = \iiint_E dv = \iint_{D_{xz}} \int_{-1}^{4-z} dy\, dA$$



$$= \iint_{D_{xz}} (4-z+1)\, dA$$

Using polar coords

$x = r\cos\theta$
$z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 (4-r\sin\theta+1)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (5r - r^2\sin\theta)\,dr\,d\theta$$

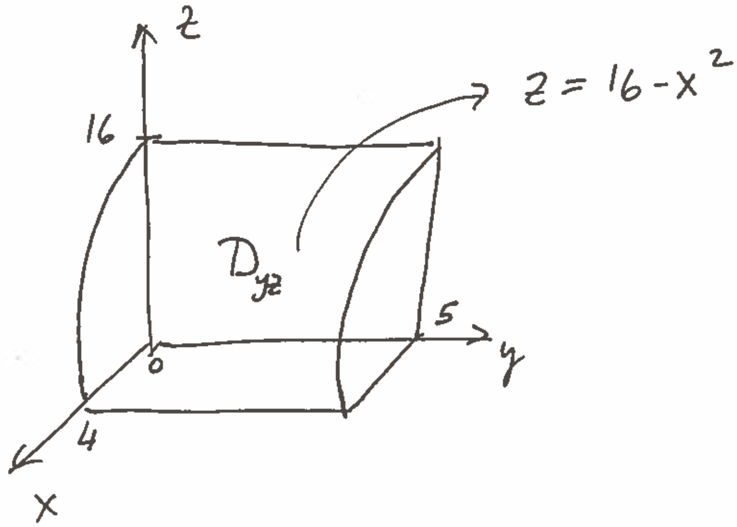
$$= \int_0^{2\pi} \left(\frac{5}{2}r^2 - \frac{r^3}{3}\sin\theta\right)\Big|_0^2 d\theta = \int_0^{2\pi}\left(10 - \frac{8}{3}\sin\theta\right)d\theta$$

$$= \left(10\theta + \frac{8}{3}\cos\theta\right)\Big|_0^{2\pi} = 20\pi + \left(\frac{8}{3}(1) - \frac{8}{3}(1)\right) = \underline{\underline{20\pi}}.$$

Previous problem in 15.2

Vol. = ?

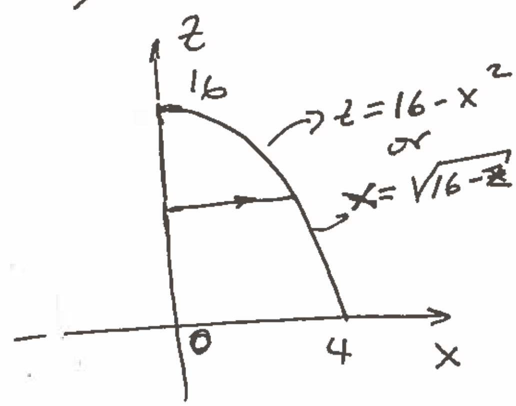
E is bded by $z = 16 - x^2$ in the first octant.
and plane $y = 5$



— Done as Type I. (3D). (xy-plane projection).

— As Type II (yz-plane projection)

$$\text{Vol} = \iint_{D_{yz}} \int_{x=0}^{\sqrt{16-z}} dx\, dA$$

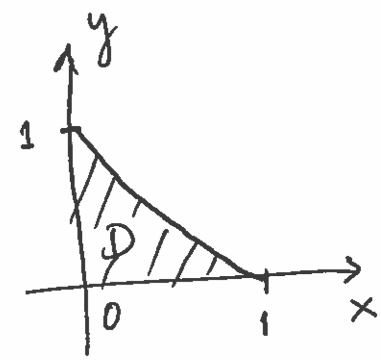


$$= \int_{y=0}^{5} \int_{z=0}^{16} \int_{x=0}^{\sqrt{16-z}} dx\, dz\, dy = \cdots = \left(\frac{128}{3}\right)^{5}.$$

15.6 #34)

Integral over E as a Type I region:

$$\iint_{\underset{(x,y)}{D}} \left(\int_{\underset{z}{0}}^{1-x^2} f\,dz\right) dA = \int_{\underset{x}{0}}^{1}\int_{\underset{y}{0}}^{1-x}\int_{\underset{z}{0}}^{1-x^2} f\,dz\,dy\,dx$$



Integral over E as a type III region

$$\iint_{\underset{(x,z)\text{ plane}}{D}} \left(\int_{\underset{y}{0}}^{1-x} f\,dy\right) dA =$$

$$= \int_{\underset{x}{0}}^{1}\int_{\underset{z}{0}}^{1-x^2}\int_{\underset{y}{0}}^{1-x} f\,dy\,dz\,dx.$$